ON A FORMULA FOR THE KOSTKA NUMBERS

M. LEDERER

ABSTRACT. By Kostant's multiplicity formula in the case of general linear groups, one can derive a formula for the Kostka numbers. In this note we give a combinatorial proof for this formula.

1. INTRODUCTION

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a partition of the integer $n \geq 1$, i.e., a sequence of integers $\alpha_i \geq 0$ such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k > 0$, $\alpha_i = 0$ for $i > k$, and $\sum_{i=1}^{k} \alpha_i = n$. Usually, we will write $\alpha = (\alpha_1, \ldots, \alpha_k)$. The *Young diagram* of α is an array of n boxes, all of the same size and aligned to each other, with k left-justified lines, where the i-th line contains α_i boxes, for all $i = 1, \ldots, k$. Let $\beta = (\beta_1, \beta_2, \ldots)$ be a vector of nonnegative integers with cumulative sum $\sum_i \beta_i = n$. If there exists an integer ℓ satisfying $\beta_\ell > 0$ and $\beta_i = 0$ for $i > \ell$, then we will write $\beta = (\beta_1, \ldots, \beta_\ell)$. A *semistandard Young tableau of shape* α *and of content* β (or *generalized Young tableau of shape* α *and of content* β) is an array of numbers which is obtained from the Young diagram of α by inserting the number i into β_i boxes , for all i, such that

- the entries in the rows of the diagram are weakly increasing, and
- the entries in the columns of the diagram are strictly increasing.

Usually one leaves the boxes away after having filled the diagram with numbers. Here is an example of a semistandard Young tableau of shape $\alpha = (4, 4, 3, 3)$ and of content $\beta = (3, 3, 2, 2, 3, 1)$:

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & \\ 5 & 5 & 6 & \end{array}$$

Semistandard Young tableaux appear in various branches of mathematics, see the survey article [10] for an overview. In many cases, a certain set of Young tableaux, subject to some constraint, has to be counted, see, for example, MacMahon [9, Vol. 2, Section 429], Knuth [4], Gordon [3], Krattenthaler [6,7].

Let E be the set of all finitely supported sequences with integer values. Let $s\colon E \to \mathbb{Z}$ be the function that sums the components of an element of E; we will need the element $0 := (0, 0, \ldots)$ of E; we will need, for all $k \in \mathbb{N}$, the element $(k) := (1, 2, \ldots, k, 0, \ldots)$ of E and we will need, for all $k \in \mathbb{N}$ and for all $\sigma \in S_k$, the element $(\sigma(k)) := (\sigma(1), \sigma(2), \ldots, \sigma(k), 0, \ldots)$ of E. We will also need

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sums and differences of elements of E. In other words, E will have to carry a $\mathbb{Z}$-module structure. For our purposes, the right way to define addition and scalar multiplication is the most natural way, i.e., componentwise. Further, for every sequence $\rho = (\rho_1, \rho_2, \ldots)$ of integers, we define E_ρ to be the set of all elements δ of E such that $s(\delta) = 0$, or $s(\delta) = \rho_1$, or $s(\delta) = \rho_1 + \rho_2$, etc. Finally, let E_+ be the set of sequences in E with nonnegative values.

Definition 1.1. *Define a map* $\mu_\rho \colon E_\rho \to \mathbb{N}_0$ *by the three properties*

- $\mu_\rho(0) = 1$,
- $\mu_\rho(\delta) = 0$ *if some* $\delta_i < 0$,
- $\mu_\rho(\delta) = \sum_{\gamma \in E_+,\, s(\gamma) = \rho_\ell} \mu_\rho(\delta - \gamma)$ *if* $s(\delta) = \rho_1 + \cdots + \rho_\ell$.

Extend the domain of definition of μ_ρ *by setting* $\mu_\rho(\delta) := 0$ *for all* $\delta \notin E_\rho$.

Let us give some of the values of μ_ρ in the example $\rho = (3, 2, 3, 2, \ldots)$. We compute some of the values of $\mu_\rho(\delta)$ for sequences of the form $\delta = (\delta_1, \delta_2, 0, \ldots)$. In the following matrix, we insert $\mu_\rho(\delta)$ at the position (δ_1, δ_2).

1	0	0	1	0	1	0	0	1
0	0	1	0	2	0	0	3	0
0	1	0	3	0	0	6	0	10
1	0	3	0	0	9	0	18	0
0	2	0	0	10	0	25	0	0
1	0	0	9	0	28	0	0	81
0	0	6	0	25	0	0	96	0
0	3	0	18	0	0	96	0	273
1	0	10	0	0	81	0	273	0

The matrix should be considered as having infinite size. Towards the left and upwards, all entries of the matrix are zero, since $\mu_\rho(\delta) = 0$ if some $\delta_i < 0$.

Let α be a partition of n and β a vector with cumulative sum n. Then define $K_{\alpha,\beta}$ to be the number of semistandard Young tableaux of shape α and of content β. The numbers $K_{\alpha,\beta}$ are called *Kostka numbers* in the literature ([11, p. 311]). It is well known that for general linear groups the dimension of the β weight-space in the irreducible highest weight module of highest weight α is equal to $K_{\alpha,\beta}$, see [1, p. 121]. From Kostant's multiplicity formula (see [2, p. 419–424] or [5]), one can easily derive the following theorem.

Theorem 1.2. *The number* $K_{\alpha,\beta}$ *of semistandard Young tableaux of shape* α *and of content* β *is given by*

$$K_{\alpha,\beta} = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \mu_\beta\big(\alpha - (k) + (\sigma(k))\big). \tag{1}$$

This result can also be obtained from the Jacobi-Trudi formula of the Schur function by observing that the number $\mu_\rho(\delta)$ equals the number of nonnegative integer matrices A satisfying $\operatorname{row}(A) = \rho$ and $\operatorname{col}(A) = \delta$, where

$$\operatorname{row}(A) = \Big(\sum_j a_{1,j}, \sum_j a_{2,j}, \ldots\Big)\ , \text{ and } \operatorname{col}(A) = \Big(\sum_i a_{i,1}, \sum_i a_{i,2}, \ldots\Big)\,.$$

In this note we present a proof of Theorem 1.2 which is both elementary, thus avoiding any theoretical apparatus, and self-contained, thus avoiding any citation of the literature.

2. A functional equation for K

The aim of this section is to prove a recursion relation for the Kostka numbers which will play a crucial role in our proof of Theorem 1.2. This recursion relation is well-known, see [8, p. 72], thus a reader who is familiar with the literature may skip this section. Nevertheless, we decided to give our own proof of the recursion relation in order to make the present note self-contained as announced above.

We would like to determine the number $K_{\alpha,\beta}$, which counts how many ways there are of filling the Young diagram of α with numbers $1,\ldots,\ell$ such that the result is a semistandard Young tableau of shape α and of content β. Let us try to fill the Young diagram of α little by little, starting with large numbers and then going downwards. We now make the first and decisive step: We insert the highest number, i.e. ℓ, precisely β_ℓ times into the Young diagram of α. From this step, we will derive the desired functional equation for f.

The obvious question is: Which possibilities are there of inserting the number ℓ precisely β_ℓ times into the Young diagram of α such that the boxes remaining unfilled can be filled with β_1 times 1, β_2 times 2, and so on, all the way up to $\beta_{\ell-1}$ times $\ell-1$, yielding a semistandard Young tableau of shape α and of content β?

In order to answer this question, let us take a look at a potential result of the filling process. More concretely, let us take a look at a fixed semistandard Young tableau of shape α and of content β. The entries in the rows of the semistandard Young tableau are increasing. This imposes the following necessary condition on the position of ℓ: No lesser number than ℓ may be inserted to the right of an ℓ in any row of the semistandard Young tableau. In other words, the numbers ℓ appearing in a fixed row of the semistandard Young diagram are aligned at the right-hand end of the row. For all $i\in\{1,\ldots,k\}$, let γ_i denote the number of occurrences of ℓ in the i-th row of the semistandard Young tableau. Put $\gamma=(\gamma_1,\gamma_2,\ldots)$. Then γ is clearly an element of E_+, and $s(\gamma)=\beta_\ell$.

So far we have used the constraint on the rows of the semistandard Young tableau. The constraint on the columns of the semistandard Young tableau implies a constraint on γ, which we formulate as a lemma.

Lemma 2.1. *For all $i\in\{1,\ldots,k\}$, we have*

$$\alpha_i-\gamma_i\geq\alpha_{i+1}\,. \tag{2}$$

Proof. Let us take a closer look at the lines number i and number $i+1$ of our fixed semistandard Young tableau of shape α and of content β. Since we are interested only in the question where ℓ is inserted, we replace every number different from ℓ by an asterisk. The result looks as follows:

$$\begin{matrix} * & * & * & * & * & \ell & \ell \\ * & * & * & * & \ell & & \end{matrix}$$

Since the entries of a semistandard Young tableau are strictly increasing, a picture like

$$\begin{matrix} * & * & * & * & \ell & \ell & \ell \\ * & * & * & * & * & & \end{matrix}$$

or

$$\begin{matrix} * & * & * & * & \ell & \ell & \ell \\ * & * & * & * & \ell & & \end{matrix}$$

does not occur. The abstract meaning of these three pictures is that the number of asterisks in the i-th row may not exceed the number of entries in the $(i+1)$-st row. Equation (2) translates this fact to a formula. □

It is important to note that (2) is equivalent to $\alpha-\gamma$ being a partition of $n-\beta_\ell$.

Proposition 2.2. *Let α and β be defined as above. Let $\beta' := (\beta_1, \ldots, \beta_{\ell-1}, 0, \ldots)$. Then the function K is subject to the following functional equation:*

$$K_{\alpha,\beta} = \sum_{\substack{\gamma\in E_+, s(\gamma)=\beta_\ell\,,\\ \forall i,\, \alpha_i-\gamma_i\geq\alpha_{i+1}}} K_{\alpha-\gamma,\beta'}\,. \tag{3}$$

Proof. Lemma 2.1 tells us that condition (2) is necessary in the following sense: For every semistandard Young tableau of shape α and of content β, there exists a $\gamma \in E_+$ satisfying (2) such that the given semistandard Young tableau of shape α and of content β is obtained from a semistandard Young tableau of shape $\alpha-\gamma$ and of content β' by the following process: Take the semistandard Young tableau of shape $\alpha-\gamma$ and of content β' and append γ_i times the number ℓ to the i-th row, for all $i \in \{1, \ldots, k\}$. It is clear that for a given semistandard Young tableau of shape α and of content β, the semistandard Young tableau of shape $\alpha-\gamma$ and of content β' which under this process leads to the given semistandard Young tableau of shape α and of content β, is unique (since, conversely, the semistandard Young tableau of shape $\alpha-\gamma$ and of content β' is obtained by erasing γ_i times the number ℓ in every line).

In an analogous sense, condition (2) is also sufficient. Let us express sufficiency as follows: Suppose to be given, along with α and β, an element γ of E_+ satisfying (2). Then for every semistandard Young tableau of shape $\alpha-\gamma$ and of content β', there exists a unique semistandard Young tableau of shape α and of content β which is obtained from the given semistandard Young tableau of shape $\alpha-\gamma$ and of content β' by appending γ_i times the number ℓ to the i-th row, for all $i \in \{1, \ldots, k\}$.

Hence there is a bijection between the set of semistandard Young tableaux of shape α and of content β, and the set of semistandard Young tableaux of shape $\alpha-\gamma$ and of content β' such that γ satisfies (2). Counting both sets gives functional equation (3). □

3. A Combinatorial Proof of the Theorem

Before stating the proof, let us rule out one potential obstruction. A given vector $\beta = (\beta_1, \ldots, \beta_\ell)$ with cumulative sum n may contain some components $\beta_i = 0$, for $i < \ell$. At first sight, this seems to cause difficulty when determining $K_{\alpha,\beta}$. Yet, it does not do so, since we can get rid of the "gaps" in β by the following process: Given β, we remove from β all $\beta_i = 0$, where $i < \ell$, push the remaining components of β to the left and call the result ρ. This is a sequence with the same nonzero components as β, and the nonzero components of ρ appear in the same order as the nonzero components of β. For the time being, let B denote the set of subscripts of the nonzero elements of β, let R denote the set of subscripts of the nonzero elements of ρ, and let t denote the unique strictly monotonous bijection $t\colon R \to B$. Then the entries of a semistandard Young tableau of shape α and of content β are clearly $\beta_{t(1)}$ times the number $t(1)$, $\beta_{t(2)}$ times the number $t(2)$, and so on, all the way up to $\beta_{t(|R|)}$ times the number $t(|R|)$. Given a semistandard Young tableau of shape α and of content β, we can replace each $t(i)$ by i. The result will be a semistandard

Young tableau of shape α and of content ρ. (The fact that the resulting array of numbers satisfies the monotony conditions for a semistandard Young tableau follows from the strict monotony of t.) Conversely, given a semistandard Young tableau of shape α and of content ρ, we can replace each i by $t(i)$ and thus obtain a semistandard Young tableau of shape α and of content β. (Again, we use the strict monotony of t.) Thus $K_{\alpha,\beta} = K_{\alpha,\rho}$. Therefore, for the rest of the paper, we can assume without loss of generality that the vector $\beta = (\beta_1, \ldots, \beta_\ell)$ contains no components $\beta_i = 0$, for $i \leq \ell$.

Proof of Theorem 1.2. Let us denote the function on the right-hand side of (1) (a function taking arguments α and β) by $g_{\alpha,\beta}$. Obviously $g_{\alpha,\beta}$ is defined for all $(\alpha, \beta) \in E \times E$, unlike $K_{\alpha,\beta}$, which is defined only for those (α, β) in $E \times E$ for which α is a partition of n and β is a vector with cumulative sum n, for some $n \geq 1$. We have to prove that $K_{\alpha,\beta} = g_{\alpha,\beta}$ for all (α, β) for which K is defined. We will do so by first showing that $g_{\alpha,\beta}$ also satisfies functional equation (3) and then showing that $K_{\alpha,\beta}$ and $g_{\alpha,\beta}$ satisfy the same boundary condition, in a sense that will be explained more precisely later.

From the definition of the function μ_β it follows that

$$g_{\alpha,\beta} = \sum_{\substack{\gamma \in E_+ \\ s(\gamma) = \beta_\ell}} g_{\alpha-\gamma,\beta}\,.$$

When computing $g_{\alpha-\gamma,\beta}$, we sum over certain $\mu_\beta(a)$, where $a \in E_\beta$ and $s(a) = \beta_1 + \cdots + \beta_{\ell-1}$. From the definition of the function μ_β it follows that for computing of $\mu_\beta(a)$, where $s(a) = \beta_1 + \cdots + \beta_{\ell-1}$, only the first $\ell - 1$ terms of β are relevant. As before, let us write $\beta' = (\beta_1, \ldots, \beta_{\ell-1}, 0, \ldots)$. Then clearly $\mu_\beta(a) = \mu_{\beta'}(a)$ for all $a \in E_\beta$ with $s(a) = \beta_1 + \cdots + \beta_{\ell-1}$. This implies that

$$g_{\alpha,\beta} = \sum_{\substack{\gamma \in E_+ \\ s(\gamma) = \beta_\ell}} g_{\alpha-\gamma,\beta'}\,. \tag{4}$$

Equation (4) is already very similar to equation (3). The difference is that in (4), the sum is taken over a larger set than in (3). Let us denote by X the difference between the index sets of the sums in (4) and in (3). Thus X is the set of all $\gamma \in E_+$ such that $s(\gamma) = \beta_\ell$, but $\alpha_i - \gamma_i \geq \alpha_{i+1}$ for all i does not hold. We will now prove the following assertion:

$$\sum_{\gamma \in X} g_{\alpha-\gamma,\beta'} = 0. \tag{5}$$

Let us define a map $\xi\colon X \to X$ in the following way: Given $\gamma \in X$, look for the smallest i such that $\alpha_i - \gamma_i < \alpha_{i+1}$. Then set

$$\xi(\gamma)_j = \begin{cases} \gamma_j, & \text{for} \quad j \neq i, i+1\,, \\ \alpha_i - \alpha_{i+1} + \gamma_{i+1} + 1, & \text{for} \quad j = i\,, \\ \alpha_{i+1} - \alpha_i + \gamma_i - 1, & \text{for} \quad j = i+1\,. \end{cases}$$

It is clear that $\xi(\gamma)$ lies in E_+ and that $s(\xi(\gamma)) = \beta_\ell$. Since $\alpha_i - \xi(\gamma)_i = \alpha_i - (\alpha_i - \alpha_{i+1} + \gamma_{i+1} + 1) < \alpha_{i+1}$, the sequence $\xi(\gamma)$ does not satisfy $\alpha_i - \gamma_i \geq \alpha_{i+1}$ for all i. Thus $\xi\colon X \to X$ is indeed well defined.

Now γ and $\xi(\gamma)$ differ only in the i-th and in the $(i+1)$-st component. For the forthcoming discussion, let us fix the transposition $\tau = (i\ i{+}1)$. Let us compare the i-th and the $(i{+}1)$-st component of $\alpha-\gamma-(k)+(\sigma(k))$ and $\alpha-\xi(\gamma)-(k)+(\sigma\circ\tau(k))$. On the one hand, we have

$$\begin{aligned}(6)\qquad &\big(\alpha-\gamma-(k)+(\sigma(k))\big)_i = \alpha_i-\gamma_i-i+\sigma(i)\\ &= \alpha_{i+1}-\xi(\gamma)_{i+1}-(i+1)+\sigma(i) = \big(\alpha-\xi(\gamma)-(k)+(\sigma\circ\tau(k))\big)_{i+1},\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}(7)\qquad &\big(\alpha-\gamma-(k)+(\sigma(k))\big)_{i+1} = \alpha_{i+1}-\gamma_{i+1}-(i+1)+\sigma(i+1)\\ &= \alpha_i-\xi(\gamma)_i-i+\sigma(i+1) = \big(\alpha-\xi(\gamma)-(k)+(\sigma\circ\tau(k))\big)_i.\end{aligned}$$

Further, for all $j\neq i,\ i+1$, we clearly have

$$(8)\qquad \big(\alpha-\gamma-(k)+(\sigma(k))\big)_j = \big(\alpha-\xi(\gamma)-(k)+(\sigma\circ\tau(k))\big)_j.$$

Equations (6), (7), (8), together with the fact that μ_β is a symmetric function, imply that

$$(9)\qquad \mu_{\beta'}\big(\alpha-\gamma-(k)+(\sigma(k))\big) = \mu_{\beta'}\big(\alpha-\xi(\gamma)-(k)+(\sigma\circ\tau(k))\big).$$

The map $S_k\to S_k\colon \sigma\mapsto\sigma\circ\tau$ is a bijection. Clearly, $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma\circ\tau)$. Therefore (9), along with the definition of $g_{\alpha,\beta}$, yields $g_{\alpha-\gamma,\beta'} = -g_{\alpha-\xi(\gamma),\beta'}$ for the case that the partition $\alpha-\gamma$ consists of k nonzero parts. If the partition $\alpha-\gamma$ consists of less than k nonzero parts, the above discussion translates literally to the situation where every k is replaced by the number of nonzero parts in the partition $\alpha-\gamma$. Thus $g_{\alpha-\gamma,\beta'} = -g_{\alpha-\xi(\gamma),\beta'}$ for all $\gamma\in X$. Since $\xi\colon X\to X$ is a bijection, it follows that $\sum_{\gamma\in X} g_{\alpha-\gamma,\beta'} = -\sum_{\gamma\in X} g_{\alpha-\gamma,\beta'}$, hence $\sum_{\gamma\in X} g_{\alpha-\gamma,\beta'} = 0$, as claimed.

Therefore, the function $g_{\alpha,\beta}$ also satisfies functional equation (3). Applying the functional equation several times, for both $K_{\alpha,\beta}$ and $g_{\alpha,\beta}$, we finally arrive at a point where the vector β in the second argument has only gotten one nonzero component, i.e., $\beta = (\beta_1, 0, \ldots)$. Hence it suffices to show that for this particular β, we have $K_{\alpha,\beta} = g_{\alpha,\beta}$, for all partitions α. This is the boundary condition for $K_{\alpha,\beta}$ and $g_{\alpha,\beta}$, announced already at the beginning of the proof. Therefore, for the rest of the proof we make the following assumptions: n is arbitrary, $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a partition of n and $\beta = (n, 0, \ldots)$.

If we try to construct semistandard Young tableaux of shape α and of content β, we have to fill the Young diagram of α with n times the number 1 such that, in particular, the entries in every column are strictly increasing. This is only possible if the Young diagram of α has just one row, i.e., if $\alpha=\beta$. In this case, there is a unique semistandard Young tableau of shape α and of content β. Thus $K_{\beta,\beta}=1$, and all $K_{\alpha,\beta}=0$ for $\alpha\neq\beta$. In other words, $K_{\beta,\beta}=1$ if $k=1$ and $K_{\alpha,\beta}=0$ if $k\geq 2$. We have to show that this is also true for $g_{\alpha,\beta}$.

Let us treat the two cases $k=1$ and $k\geq 2$ separately. For $k=1$, the only possibility for α to be a partition of n is the case $\alpha=(n,0,\ldots)$, i.e., $\alpha=\beta$. In this case, the only summand occurring in the sum defining $g_{\alpha,\beta}$ is the summand for $\sigma=\mathrm{id}$, and this summand yields $g_{\alpha,\beta}=\mu_\beta(\beta)=1$ by the definition of μ_β. For $k\geq 2$, we have to show that $g_{\alpha,\beta}=0$. In order to determine $g_{\alpha,\beta}$, we have to determine $\mu_\beta\big(\alpha-(k)+(\sigma(k))\big)$ for all $\sigma\in S_k$. For a given $\sigma\in S_k$, we distinguish between the following two cases: Either there is some i such that $\alpha_i-i+\sigma(i)<0$, in this case $\mu_\beta\big(\alpha-(k)+(\sigma(k))\big)=0$, or all $\alpha_i-i+\sigma(i)\geq 0$, then $\mu_\beta\big(\alpha - (k) + (\sigma(k))\big) = 1$. Let Y be the set of those $\sigma \in S_k$ for which all $\alpha_i - i + \sigma(i) \geq 0$. Then for all $\sigma \in Y$, the summand in $g_{\alpha,\beta}$ corresponding to σ equals $\mathrm{sgn}(\sigma)$. We thus have to show that $\sum_{\sigma\in Y} \mathrm{sgn}(\sigma) = 0$. Let us do this in a way analogous to what we have done before, namely by making use of an appropriate transposition. Here it is going to be the fixed transposition $\tau = (1\ 2)$. We proceed as follows: Our first observation is that from α being a partition of n and $k \geq 2$, we get in particular that $\alpha_1 \geq \alpha_2 \geq 1$. We deduce that $\alpha_1 - 1 + \sigma(1) \geq 1$ and $\alpha_2 - 2 + \sigma(2) \geq 0$ for all $\sigma \in S_k$. Further, we also have $\alpha_1 - 1 + \sigma(2) \geq 1$ and $\alpha_2 - 2 + \sigma(1) \geq 0$ for all $\sigma \in S_k$. In particular the first two components of $\alpha - (k) + (\sigma(k))$ and $\alpha - (k) + (\sigma\circ\tau(k))$ are nonnegative. And clearly, $\alpha - (k) + (\sigma(k))$ and $\alpha - (k) + (\sigma\circ\tau(k))$ differ only in the first two components. Thus, in particular, $\alpha - (k) + (\sigma(k))$ and $\alpha - (k) + (\sigma\circ\tau(k))$ have the same negative components. Now $\mu_\beta\big(\alpha - (k) + (\sigma(k))\big)$ and $\mu_\beta\big(\alpha - (k) + (\sigma\circ\tau(k))\big)$ can both only take the values 0 or 1, depending on whether they have some negative component or not. Therefore,

$$\mu_\beta\big(\alpha - (k) + (\sigma(k))\big) = \mu_\beta\big(\alpha - (k) + (\sigma\circ\tau(k))\big)\,. \tag{10}$$

Since the map $S_k \to S_k\colon \sigma \mapsto \sigma\circ\tau$ is a bijection, and $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma\circ\tau)$, equation (10) yields

$$\sum_{\gamma\in S_k} \mathrm{sgn}(\sigma)\mu_\beta\big(\alpha - (k) + (\sigma(k))\big) = -\sum_{\gamma\in S_k} \mathrm{sgn}(\sigma)\mu_\beta\big(\alpha - (k) + (\sigma(k))\big), \tag{11}$$

hence $g_{\alpha,\beta} = 0$, as claimed. □

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Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany
E-mail address: `mlederer@math.uni-bielefeld.de`