Times of arrival: Bohm beats Kijowski

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Abstract. We prove that the Bohmian arrival time of the 1D Schrödinger evolution violates the quadratic form structure on which Kijowski's axiomatic treatment of arrival times is based. Within Kijowski's framework, for a free right moving wave packet Ψ, the various notions of arrival time (at a fixed point x on the real line) all yield the same average arrival time $\bar{t}_{K_{ij}}(\Psi)$. We derive the inequality $\bar{t}_B(\Psi) \leq \bar{t}_{K_{ij}}(\Psi)$ relating the average Bohmian arrival time to the one of Kijowksi. We prove that $\bar{t}_B(\Psi) < \bar{t}_{K_{ij}}(\Psi)$ if and only if Ψ leads to position probability backflow through x.

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1. Introduction

Let a ready particle detector be exposed to a propagating one particle wave function. What is the probability distribution of the time when the detector clicks? Even in the simplest case of a free one dimensional Schrödinger wave function, the proposed answers to this question for an "intrinsic, free arrival time distribution" remain controversial. (See e.g. the introduction to [1].) The problem arises from the fact that quantum mechanics provides probability distributions only for the outcomes of measurements performed at a certain time t, which has to be chosen by the observer. And no such choice shows up in the above situation.

Among the various notions of arrival time, offered by standard quantum mechanics, the most prominent one arises from the generalized resolution of the identity associated with the arrival time operator of Aharonov and Bohm [2]. This operator's density of arrival times also belongs to a set of arrival time densities proposed by Kijowski [3] and it is unique within this set insofar as it minimizes, for every wave function, the variance of arrival times. Kijowski determined his set from a list of axiomatic properties that seem plausible within standard quantum mechanics. A summary of these matters is given by Egusquiza *et al* in section 10 of [1].

Bohmian mechanics extracts a probability space of particle trajectories from each solution of a configuration space Schrödinger equation. This trajectory space seems a

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natural candidate from which to derive the intrinsic arrival time distribution of an arbitrary wave function. One simply has to answer the following question for any $t \in \mathbb{R}$: What is the probability measure of the subset of trajectories which intersect the detector's volume at any time s prior to time t? Leavens seems to have made the first use of this idea [4]. For certain 1D scattering wave functions ψ_t, he derived the position probability current $J_x(\psi_t)$ at point x to be identical (up to normalization) with the conditional probability density for the arrival at point x at time t. (The conditioning is made to the event that an arrival at x occurs at all.) In the 3D case the Bohmian strategy has been outlined by Daumer, Dürr, Goldstein, and Zanghi in their contribution to [5] and in [6]. Later on for the general 1D case Leavens has argued that within Bohmian mechanics $|J_x(\psi_t)|$ (up to normalization) is identical with the conditional probability density for the arrival at point x at time t [7]. While Leavens' argument is correct under certain limited circumstances it is wrong in general. A cutoff procedure for reentering trajectories is missing from $|J_x(\psi_t)|$ [6, 8]. Instead of this, as has been shown in [9], a more complicated expression, involving the current $J_x(\psi_s)$ at all times s prior to t, yields, within Bohmian mechanics, the conditional probability density for the arrival at point x at time t.

In the present work we study the question whether the Bohmian arrival time density, restricted to free 1D positive momentum wave functions, belongs to the set of arrival time densities introduced axiomatically by Kijowski. We shall prove that it does not do so, since already the basic quadratic form structure, which Kijowski assumes, is violated. No wonder that the expectation values of arrival times according to Bohm and according to Kijowski in general differ. We shall show that the Bohmian expectation value is less or equal to the one according to Kijowski. And it is exactly for wave functions without position space probability backflow through the arrival point x that the two expectation values coincide. (A 1D positive momentum wave function whose position probability current at the arrival point x takes on negative values during a finite time interval is said to be a wave function with backflow. For such a wave function the detection probability on the half line right of x is not monotonically increasing from 0 to 1 as a function of time.)

This leads us to the question of measurability of Bohmian arrival times. As we understand it, the main virtue of Bohmian mechanics with its introduction of a definite position in configuration space, is the fact that it provides the mathematical structure to represent within quantum theory the empirical fact that individual systems have properties. In this manner Bohmian mechanics gets rid of the quantum measurement problem. But it does so only if it is assumed that a system's properties, which may encompass an observer's perception, are completely determined by its Bohmian configuration. (Unlike wave functions, Bohmian positions are definite and unsplit.) Therefore it seems likely that a detection event happens as soon as a sufficient change in the detector's (or the observers) Bohmian configuration has taken place. This happens at about the instant, when the Bohmian position of the detected particle passes the detector. Why? Because the 'empty' partial waves, hitting the detector, indeed change the detector's wave function, but their dynamical relevance to the detector's Bohmian position is negligible. Thus, according to this picture, it should be the Bohmian arrival times, which show up in time resolved detection experiments.

Sections 2 and 3 summarize the basic facts about arrival time densities according to Kijowski and Bohm. In section 4 we proof two theorems relating these two notions. The moral of our study is distilled in section 5. A concise review of Bohmian mechanics can be found in [10].

2. Kijowski's arrival time densities

In 1974 Kijowski [3] introduced a set of conceivable quantum mechanical arrival time probability densities for a subspace of right moving wave functions. These densities are parametrized by quadratic forms of a certain type. We shall describe them in what follows.

Definition 1 *Let* $\mathcal{D}$ *be a complex vector space. A function* $q : \mathcal{D} \to \mathbb{R}$ *is called a quadratic form, if there exists a hermitian sesquilinear form* $S : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ *such that* $q(\phi) = S(\phi, \phi)$ *for all* $\phi \in \mathcal{D}$.

Definition 2 *Let* $\mathcal{D}(\mathbb{R}_+)$ *denote the space of test functions with compact support in* $\mathbb{R}_+ :=]0, \infty[$ *with the usual notion of convergence. Then*

$$\phi \mapsto \phi_t \textit{ with } \phi_t(k) = \exp\left(-(\mathrm{i}\hbar k^2 t/2m)\right)\phi(k) \textit{ for } t \in \mathbb{R}$$

gives the free Schrödinger time evolution. Let $\mathcal{Q}$ *denote the set of all continuous quadratic forms* $q : \mathcal{D}(\mathbb{R}_+) \to \mathbb{R}$ *such that for all* $\phi \in \mathcal{D}(\mathbb{R}_+)$ *holds*

(i) $q(\phi) \geq 0$,
(ii) $q(\overline{\phi}) = q(\phi)$,
(iii) $\int_{-\infty}^{\infty} q(\phi_t)\, \mathrm{d}t = \|\phi\|^2$,
(iv) $\overline{t^2}(q, \phi) := \int_{-\infty}^{\infty} t^2 q(\phi_t)\, \mathrm{d}t < \infty$.

For any $q \in \mathcal{Q}$ *the non-negative function* $D_{\phi,q} : \mathbb{R} \to \mathbb{R}, t \mapsto q(\phi_t)$ *yields a conceivable arrival time density at* $x = 0$ *for the wave function* $\phi \in \mathcal{D}(\mathbb{R}_+)$ *subject to* $\|\phi\| = 1$. *The case of arbitrary* $x \in \mathbb{R}$ *is obtained by replacing* ϕ *in* $D_{\phi,q}$ *with the function* $k \mapsto e^{-ikx}\phi(k)$.

According to (iv), for all $q \in \mathcal{Q}$ the second moment of the density $D_{\phi,q}$ is finite. Due to the continuity of q also the first moment

$$\overline{t}(q, \phi) := \int_{-\infty}^{\infty} t\, q(\phi_t)\, \mathrm{d}t$$

is finite. The variance of arrival times is given by $V(q, \phi) := \overline{t^2}(q, \phi) - (\overline{t}(q, \phi))^2$. The quadratic form $q_0 : \mathcal{D}(\mathbb{R}_+) \to \mathbb{R}$

$$q_0(\phi) := \frac{\hbar}{(2\pi)m} \left| \int_{-\infty}^{\infty} \sqrt{k}\, \phi(k)\, \mathrm{d}k \right|^2,$$

belongs to $\mathcal{Q}$. The probability density D_{ϕ,q_0} is equal to the arrival time density derived from the Aharonov-Bohm arrival time operator and it is distinguished by the following theorem.

Theorem 1 (Uniqueness Theorem) *For all* $q \in \mathcal{Q}$ *and for all* $\phi \in \mathcal{D}(\mathbb{R}_+)$ *with* $\|\phi\| = 1$ *there holds*

(i) $\bar{t}(q_0, \phi) = \bar{t}(q, \phi)$ *and*

(ii) $V(q_0, \phi) \leq V(q, \phi)$.

Furthermore $V(q, \phi) = V(q_0, \phi)$ *for all* $\phi \in \mathcal{D}(\mathbb{R}_+)$ *with* $\|\phi\| = 1$ *if and only if* $q = q_0$.

The proof of this theorem is to be found in [3].

3. Bohmian arrival time density

Let $x \mapsto \Phi_t(x) := 1/\sqrt{2\pi}\int_{-\infty}^{\infty} \exp(\mathrm{i}kx)\phi_t(k)\,\mathrm{d}k$ denote the freely evolving configuration space wave function at time t associated with the momentum space wave function $\phi \in \mathcal{D}(\mathbb{R})\backslash 0$. Let $P_\phi(t)$ denote the probability measure of the set of this wave function's Bohmian trajectories which cross $x = 0$ at some time $s \in]-\infty, t]$. Again the arrival at arbitrary $x \in \mathbb{R}$ is obtained by replacing ϕ in P_ϕ with the function $k \mapsto e^{-ikx}\phi(k)$. (If we assume an ideal detector to be placed at $x = 0$, then, according to Bohmian mechanics, $P_\phi(t)$ is equal to the detection probability of that wave function at any time $s \in]-\infty, t]$.) Let

$$J_x(\phi) := \frac{\hbar}{2m\mathrm{i}}(\overline{\Phi(x)}\frac{\partial}{\partial x}\Phi(x) - \Phi(x)\frac{\partial}{\partial x}\overline{\Phi(x)})$$

denote this wave function's probability current at $t = 0$ and at position x. It follows that

$$J_0(\phi) = \frac{\hbar}{2m}\left\{\frac{1}{(2\pi)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(k + l)\,\overline{\phi(l)}\phi(k)\;\mathrm{d}k\,\mathrm{d}l\right\}.$$

Then the following two theorems hold [9]:

Theorem 2 *Let* $\phi \in \mathcal{D}(\mathbb{R})$ *with* $\|\phi\| = 1$. *Then for all* $t \in \mathbb{R}$

$$P_\phi(t) = \sup\{f_\phi(s)| - \infty < s \leq t\} + \sup\{-f_\phi(s)| - \infty < s \leq t\} \tag{1}$$

with

$$f_\phi(t) := \int_{-\infty}^{t} J_0(\phi_s)\,\mathrm{d}s.$$

From the detection probability one can define in the usual way a conditional arrival time probability density $B_\phi : \mathbb{R} \to \mathbb{R}$ by

$$\frac{P_\phi(t)}{\lim_{s\to\infty} P_\phi(s)} = \int_{-\infty}^{t} B_\phi(s)\,\mathrm{d}s.$$

Theorem 3 *For* $\phi \in \mathcal{D}(\mathbb{R})$ *and* $\|\phi\| = 1$ *holds*

$$B_\phi(t) = \Big(\lim_{s\to\infty} P_\phi(s)\Big)^{-1} \Big[J_0(\phi_t)\cdot\chi\Big(f_\phi(t) - \sup_{-\infty<s\le t}\{f_\phi(s)\}\Big) - J_0(\phi_t)\cdot\chi\Big(-f_\phi(t) - \sup_{-\infty<s\le t}\{-f_\phi(s)\}\Big)\Big] \ge 0. \tag{2}$$

Here χ *denotes the cutoff function*

$$\chi(s) = \begin{cases} 0 & for\ \ s \neq 0 \\ 1 & for\ \ s = 0. \end{cases}$$

The probability current may become negative, even for $\phi \in \mathcal{D}(\mathbb{R}_+)$, a fact which is known as the quantum backflow effect [11]. The cutoff function guaranties the non negativity of the probability density and prevents a multiple counting of trajectories.

From now on we restrict ourselves to right moving states, i.e. wave functions $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$. Due to the half-line localization of Φ_t at $x < 0$ for $t \to -\infty$ and from probability conservation we conclude

$$f_\phi(t) := \int_{-\infty}^{t} J_0(\phi_s)\,\mathrm{d}s = \int_0^\infty |\Phi_t(x)|^2\,\mathrm{d}x. \tag{3}$$

From this it follows that $0 \le f_\phi \le 1$. As $\lim_{t\to-\infty} f_\phi(t) = 0$, we have $\sup\{-f_\phi(s)|-\infty < s \le t\} = 0$ for all $t \in \mathbb{R}$. Thus $P_\phi(t)$, according to equation (1), simplifies to

$$P_\phi(t) = \sup\{f_\phi(s)|\ -\infty < s \le t\}.$$

The half-line localization of Φ_t at $x > 0$ for $t \to \infty$ implies $1 = \lim_{t\to\infty} f_\phi(t) = \lim_{t\to\infty} P_\phi(t)$. Therefore equation (2) simplifies to

$$B_\phi(t) = J_0(\phi_t)\cdot\chi\Big(f_\phi(t) - \sup_{-\infty<s\le t}\{f_\phi(s)\}\Big) \ge 0. \tag{4}$$

4. Bohm versus Kijowski

In this section we investigate the question whether the Bohmian arrival time density B_ϕ belongs to the class of arrival time densities considered by Kijowski. This is the case if and only if there exists a quadratic form $q \in \mathcal{Q}$ such that

$$B_\phi(t) = q\left(\phi_t\right)$$

for all $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$ and for all $t \in \mathbb{R}$. The following theorem demonstrates that the answer to the above question is no.

Theorem 4 *There is no quadratic form* q *on* $\mathcal{D}(\mathbb{R}_+)$ *such that* $B_\phi(0) = q(\phi)$ *for all* $\phi \in \mathcal{D}(\mathbb{R}_+)$ *with* $\|\phi\| = 1$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\varphi\| = 1$ such that $B_\varphi(0) > 0$. A second unit vector $\psi \in \mathcal{D}(\mathbb{R}_+)$ is chosen such that $J_0(\psi) < 0$. Thus we have

$$f_\psi(0) = \int_{-\infty}^{0} J_0(\psi_t)\,\mathrm{d}t < \sup_{-\infty<s\le 0}\{f_\psi(s)\}.$$

From this then follows by means of equation (4)

$$B_\psi(0) = J_0(\psi) \cdot \chi(f_\psi(0) - \sup_{-\infty < s \leq 0} \{f_\psi(s)\}) = 0. \tag{5}$$

Since $\phi \mapsto J_0(\phi)$ is a quadratic form its restriction to a 2D subspace is continuous. Thus the mapping

$$\xi \mapsto J_0(\cos(\xi)\varphi + \sin(\xi)\psi) =: j(\xi)$$

is continuous on the interval $[0, \pi/2]$. Since $j(\pi/2) = J_0(\psi) < 0$ there exists a number $\eta \in]0, \pi/2[$ such that $j(\xi) < 0$ for all $\xi \in [\eta, \pi/2]$. In consequence the mapping

$$\xi \mapsto B_{\cos(\xi)\varphi + \sin(\xi)\psi}(0) =: \beta(\xi)$$

obeys $\beta(0) = B_\varphi(0) > 0$ and $\beta(\xi) = 0$ for all $\xi \in [\eta, \pi/2]$.

Assume now that there exists a quadratic form $q \in \mathcal{D}(\mathbb{R}_+)$ such that $B_\phi(0) = q(\phi)$ for all $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$. Let S denote the hermitian sesquilinear form associated with q. Then we have

$$\beta(\xi) = \cos^2(\xi) q(\varphi) + \sin^2(\xi) q(\psi) + \sin(\xi)\cos(\xi) 2\Re(S(\varphi, \psi)). \tag{6}$$

Since $\beta(0) > 0$, equation (6) implies

$$q(\varphi) > 0. \tag{7}$$

Similarly $\beta(\pi/2) = 0$ implies $q(\psi) = 0$. Let $\epsilon \in]\eta, \pi/2[$. Then we have

$$0 = \frac{\beta(\eta)}{\cos^2(\eta)} = q(\varphi) + 2\Re(S(\varphi, \psi)) \tan(\eta)$$

and

$$0 = \frac{\beta(\epsilon)}{\cos^2(\epsilon)} = q(\varphi) + 2\Re(S(\varphi, \psi)) \tan(\epsilon).$$

Since $\epsilon \neq \eta$ and $\tan : [0, \pi/2[\to \mathbb{R}$ is injective we conclude from

$$2\Re(S(\varphi, \psi)) \cdot \tan(\epsilon) = 2\Re(S(\varphi, \psi)) \cdot \tan(\eta)$$

that $\Re(S(\varphi, \psi)) = 0$. Due to $\beta(\epsilon) = 0$ we now have $q(\varphi) = 0$ in contradiction to $q(\varphi) > 0$ (see equation (7)). ■

Now we compare the first moments of the probability densities $D_{\phi,q}$ according to Kijowski on the one side, and B_ϕ according to Bohmian mechanics on the other side. As has been shown in [3], for all $q \in \mathcal{Q}$ and for all $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$ there holds

$$\int_{-\infty}^{\infty} t J_0(\phi_t) \, \mathrm{d}t = \int_{-\infty}^{\infty} t q(\phi_t) \, \mathrm{d}t =: \bar{t}(q, \phi). \tag{8}$$

In view of the backflow effect this is somewhat surprising. The following theorem relates the first moments of $D_{\phi,q}$ and of B_ϕ. The first moment of latter density is denoted as

$$\bar{t}(B_\phi) := \int_{-\infty}^{\infty} t B_\phi(t) \, \mathrm{d}t.$$

Theorem 5 *Let ϕ be in $\mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$. Then $\bar{t}(q, \phi) \geq \bar{t}(B_\phi)$ for all $q \in \mathcal{Q}$. Equality $\bar{t}(q, \phi) = \bar{t}(B_\phi)$ holds if and only if $J_0(\phi_t) \geq 0$ for all $t \in \mathbb{R}$.*

Proof. For $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$ and with $J_0(\phi_t) \geq 0$ for all $t \in \mathbb{R}$, the function $f_\phi(t)$ is nondecreasing. Therefore, according to equation (4), there holds $J_0(\phi_t) = B_\phi(t)$ for all $t \in \mathbb{R}$. Thus from equation (8) we conclude $\bar{t}(q,\phi) = \bar{t}(B_\phi)$.

Assume now $\phi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\phi\| = 1$ and that there exists a $t \in \mathbb{R}$ such that $J_0(\phi_t) < 0$. Then the open set

$$\Delta_< := \left\{ t \in \mathbb{R} \,\middle|\, f_\phi(t) < \sup_{-\infty < s \leq t} \{f_\phi(s)\} \right\} \subset \mathbb{R}$$

is nonempty. Note that for all $t \in \mathbb{R}\backslash\Delta_<$ the equality $f_\phi(t) = \sup_{-\infty<s\leq t}\{f_\phi(s)\}$ holds. The set $\Delta_<$ is a disjoint union of open intervals $]a,b[$ such that $f_\phi(a) = f_\phi(b)$. Then we have, according to equation (8), that

$$\bar{t}(q,\phi) = \int_{\mathbb{R}\backslash\Delta_<} tJ_0(\phi_t)\mathrm{d}t + \int_{\Delta_<} tJ_0(\phi_t)\mathrm{d}t. \tag{9}$$

Equation (4) implies

$$B_\phi(t) = \left\{ \begin{array}{rl} 0 & \text{for all } t \in \Delta_< \\ J_0(\phi_t) & \text{for all } t \in \mathbb{R}\backslash\Delta_< \end{array} \right. .$$

From this and from equation (9) we infer

$$\bar{t}(q,\phi) = \int_{\mathbb{R}\backslash\Delta_<} tB_\phi(t)\mathrm{d}t + \int_{\Delta_<} tJ_0(\phi_t)\mathrm{d}t = \bar{t}(B_\phi) + \int_{\Delta_<} tJ_0(\phi_t)\mathrm{d}t.$$

The latter integral over $\Delta_<$ is a sum of integrals over disjoint intervals $]a,b[$. Denote

$$F(t) = f_\phi(t) - f_\phi(a) = \int_a^t J_0(\phi_s)\mathrm{d}s$$

then holds $F'(t) = J_0(\phi_t)$ and $F(t) < 0$ for all $t \in]a,b[$ and $F(a) = F(b) = 0$. With this each of the integrals can be estimated by means of a partial integration as follows.

$$\int_a^b tJ_0(\phi_t)\mathrm{d}t = \int_a^b tF'(t)\mathrm{d}t = tF(t)|_a^b - \int_a^b F(t)\mathrm{d}t = -\int_a^b F(t)\mathrm{d}t > 0.$$

Thus for wave functions with backflow we have $\bar{t}(q,\phi) - \bar{t}(B_\phi) = -\int_{\Delta_<} F(t)\mathrm{d}t > 0.$ ■

In a completely analogous way one can show for $\phi \in \mathcal{D}(\mathbb{R}_-)$ that $\bar{t}(q,\phi) \leq \bar{t}(B_\phi)$. See [12].

5. Conclusion

Now, as the Bohmian arrival time density, associated with a wave function $\phi \in \mathcal{D}(\mathbb{R}_+)$, does not belong to the set of arrival time densities introduced by Kijowski, one may wonder how deep this discrepancy goes. Indeed, as is obvious from our proof of Theorem (4), the arrival time density B_ϕ violates the quadratic form structure, which is considered as one of the basic rules of standard quantum mechanics. To recall this point: We have seen that for some φ, $\psi \in \mathcal{D}(\mathbb{R}_+)$ with $\|\varphi\| = \|\psi\| = 1$ the Bohmian arrival time density obeys

$$B_{\cos(\xi)\varphi+\sin(\xi)\psi}(0) \neq a + b\cos(2\xi) + c\sin(2\xi)$$

for any choice of the constants $a, b, c \in \mathbb{R}$. This fact also contradicts that version of Bohmian mechanics, which is empirically equivalent to standard quantum mechanics. The essence of that version is expressed most succinctly in proposition (2) of [10]. It is crucial for this proposition that the random variables on the configuration space, which are averaged over with the position density, are not allowed to depend on the wave function. The definition of the Bohmian arrival time density B_ϕ, however, makes use of a random variable, which parametrically depends on the wave function ϕ. Thus assuming that B_ϕ is observable implicitly generalizes the standard rules of how to model experiments and might lead to an empirical discrimination between Bohmian mechanics and standard quantum theory.

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